Lecture 21: Discrete Fourier Analysis on the Boolean Hypercube (Convolution)

Linear shift of a Function I

- Let f, g be functions such that $f(x) = g(x - c)$, for some fixed $c \in \{0,1\}^n$
- Then, the following result holds

Lemma

$$\widehat{f} = \chi_c \widehat{g}$$

What this result means is that $\widehat{f}(S) = \chi_c(S)\widehat{g}(S)$, for all $S \in \{0,1\}^n$.

Linear shift of a Function II

- Proof Outline.

$$\begin{aligned}
\widehat{f}(S) = \langle f, \chi_S \rangle &= \frac{1}{N} \sum_{x \in \{0,1\}^n} f(x)\chi_S(x) \\
&= \frac{1}{N} \sum_{x \in \{0,1\}^n} g(x-c)\chi_S(x) \\
&= \frac{1}{N} \sum_{x \in \{0,1\}^n} g(x-c)\chi_S(x-c)\chi_S(c) \\
&= \chi_S(c)\widehat{g}(S) \\
&= \chi_c(S)\widehat{g}(S)
\end{aligned}$$

- In this proof, we used two properties of the Fourier basis function $\chi_S(x+y) = \chi_S(x)\chi_S(y)$, and $\chi_S(x) = \chi_x(S)$

Linear shift of a Function III

- Basically, this result states that g and the "shifted version of g" (i.e., the function f) have closely related Fourier coefficients. We interpret $\widehat{f}(S)$ as "rotation" of $\widehat{g}(S)$ by the "phase" $\chi_c(S)$. Basically, $\widehat{f}(S)$ has the same magnitude as $\widehat{g}(S)$, except that it is "rotated" suitably.
- Think: Let A be an affine space (i.e., offset of a vector subspace of $\{0,1\}^n$). What is $\widehat{\mathbf{1}_{\{A\}}}$?

Convolution I

- Given two function $f, g\colon \{0,1\}^n \to \mathbb{R}$, we define the following function $h\colon \{0,1\}^n \to \mathbb{R}$

$$h(x) = \frac{1}{N} \sum_{y \in \{0,1\}^n} f(y) g(x - y)$$

- We say that h is the convolution of f and g. We represent the convolution of f and g as $(f * g)$.
- We have the following result

Lemma

$$\widehat{(f * g)} = \widehat{f}\widehat{g}$$

That is, we have $\widehat{(f * g)}(S) = \widehat{f}(S)\widehat{g}(S)$, for all $S \in \{0,1\}^n$.

Convolution II

- Proof outline.

$$\begin{aligned}
\widehat{(f*g)}(S) &= \langle f*g, \chi_S \rangle = \frac{1}{N}\sum_{x\in\{0,1\}^n} (f*g)(x)\chi_S(x) \\
&= \frac{1}{N}\sum_{x\in\{0,1\}^n} \frac{1}{N}\sum_{y\in\{0,1\}^n} f(y)g(x-y)\chi_S(x) \\
&= \frac{1}{N}\sum_{x\in\{0,1\}^n} \frac{1}{N}\sum_{y\in\{0,1\}^n} f(y)g(x-y)\chi_S(y)\chi_S(x-y) \\
&= \left(\frac{1}{N}\sum_{y\in\{0,1\}^n} f(y)\chi_S(y)\right)\left(\frac{1}{N}\sum_{r\in\{0,1\}^n} g(r)\chi_S(r)\right) \\
&= \widehat{f}(S)\widehat{g}(S)
\end{aligned}$$